

Embedding problem of linear fractional maps of B_N

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Abstract

This paper focuses on the embedding problem of linear fractional maps which explains when a linear fractional self-map can be a member of a semi-group of self-maps of B_N .

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1 Introduction

Let B_N be the open unit ball of \mathbb{C}^N and $H(B_N)$ the collection of holomorphic functions defined on B_N . Cowen and MacCluer discussed a class of self-map called linear fractional map based on the research of automorphism of B_N in [8]. A linear fractional map φ is given by

$$\varphi(z) = \frac{Az + B}{\langle z, C \rangle + D},$$

where $A \in \mathbb{C}^{N \times N}$, $B \in \mathbb{C}^N$, $C \in \mathbb{C}^N$ and $D \in \mathbb{C}$ satisfy several conditions. Properties of linear fractional maps have been deeply studied. We refer the reader to the excellent papers written by Cowen and MacCluer [7, 8], Bayart [1, 2], Bisi and Bracci [4], Jiang and Ouyang [9] etc.

The following theorem is the Denjoy-Wolff Theorem corresponding to the one dimensional case:([10])

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Theorem 1.1 (*Denjoy-Wolff Theorem on B_N*) Suppose that φ is a holomorphic self-map of B_N without interior fixed points. There is a unique $w \in \partial B_N$ such that the iteration series $\{\varphi_n\}$ converges to w uniformly on compact subsets of B_N .

w in the above theorem is called the Denjoy-Wolff point of φ . According to Theorem 1.3 in [10], there is a positive number $\delta \in (0, 1]$ such that

$$0 < \liminf_{z \rightarrow w} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} = \delta \leq 1,$$

δ will be defined as the boundary dilation coefficient of φ at w , or simply the boundary dilation coefficient of φ . With the help of Denjoy-Wolff point and boundary dilation coefficient, the holomorphic self-maps of B_N will be assort by the following definition:

Definition 1.2 Suppose φ is a holomorphic self-map of B_N .

1. φ is elliptic if φ has at least one interior fixed point;
2. φ is hyperbolic if φ has no interior fixed point and boundary dilation coefficient $\delta \in (0, 1)$;
3. φ is parabolic if φ has no interior fixed point and boundary dilation coefficient $\delta = 1$.

A continuous semigroup $\{\varphi_t\}$ of $H(B_N)$ in B_N is a continuous homomorphism from the additive semigroup of non-negative real numbers into the composition semigroup of all holomorphic self-maps of B_N endowed with the compact-open topology (Definition 2.6). As for theories about semigroups in the unit disc \mathbb{D} which has been deeply studied and applied in many different contexts, we refer the reader to Shoikhet [13] and references therein.

As to semigroup of linear fractional self-maps in the case of higher dimension, Bracci and co-workers [5] gave a complete classification up to conjugation of continuous semigroups of linear fractional self-maps of B_N , and in [6], they characterized the infinitesimal generator of a semigroup of linear fractional self-maps of B_N and solved the embedding problem on \mathbb{D} from a dynamical point of view.

In this paper, we considered the embedding problem for semigroups of linear fractional self-maps on B_N . This problem may be completely solved when the linear fractional self-map φ is elliptic. In the non-elliptic cases, some sufficient conditions about the embedding problem were given when

φ is normal. According to these conditions, the embedding problem when the dimension $N = 2$ could be solved. These conditions also give an answer to the embedding problem when the linear fractional self-map was an automorphism.

2 Background material

Suppose that $A \in \mathbb{C}^{N \times N}$, the spectrum of A will be denoted by $\sigma(A)$, and the spectra radius

$$\rho(A) = \max \{ |\lambda| : \lambda \in \sigma(A) \}.$$

Let A^H be the conjugate transpose of A , and $\|A\|$ be the spectra norm of A , i.e.

$$\|A\| = \rho(A^H A)^{\frac{1}{2}} = \sup_{x \in \mathbb{C}^N, |x|=1} |Ax|.$$

Given a matrix $A \in \mathbb{C}^{N \times N}$,

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

may be defined as the exponential of A , where $A^0 = E$, $A^n = A \cdots A$, n times repeated, for $n \geq 1$. The exponential of a matrix will always be defined.

There is a classic proposition about matrix analysis. ([3], P241, Theorem 71)

Lemma 2.1 *Given any invertible matrix $A \in \mathbb{C}^{N \times N}$, there exist a matrix M such that $\exp(2\pi i M) = A$. If M is triangularly blocked of some type, then so are the matrices M . The eigenvalues of any two such M can only differ by integers, and there is a unique matrix M whose eigenvalues have real parts in the half-open interval $[0, 1)$.*

Definition 2.2 *Let $M \in \mathbb{C}^{N \times N}$, we say that M is dissipative if for any $w \in \mathbb{C}^N$,*

$$\operatorname{Re} w^H M w \leq 0.$$

Proposition 2.3 *(Phillips-Lumer, [5]) For any $t \in \mathbf{R}$, $\|\exp(tM)\| \leq 1$ if and only if M is dissipative.*

It would be easier to verify whether M is dissipative or not in the case that M is normal.

Proposition 2.4 *Let $M \in \mathbb{C}^{N \times N}$ be normal. Then M is dissipative if and only if $\|\exp(M)\| \leq 1$.*

Proof. Suppose firstly that $\|\exp(M)\| \leq 1$. Based on $MM^H = M^H M$,

$$\exp(M) \exp(M)^H = \exp(M + M^H).$$

Since

$$\|\exp(M)\|^2 = \rho(\exp(M) \exp(M)^H),$$

for any $\lambda \in \sigma(M + M^H)$, we see that

$$|\exp(\lambda)| \leq 1,$$

therefore, $\operatorname{Re} \lambda \leq 0$. Further more, $\lambda \leq 0$ results from that $M + M^H$ is a hermitian matrix. As a consequence, for any $w \in \mathbb{C}^N$, we get

$$w^H (M + M^H) w \leq 0,$$

or

$$\operatorname{Re} w^H M w \leq 0.$$

Therefore M is dissipative.

Otherwise, if M is dissipative, for any $t \in \mathbf{R}$,

$$\|\exp(tM)\| \leq 1$$

according to Phillips-Lumer's theorem. Let $t = 1$, we obtain

$$\|\exp(M)\| \leq 1.$$

■

We denoted by $LFT(B_N)$ the set of linear fractional self-maps of B_N . Let V be an one dimensional affine subset of \mathbb{C}^N and let

$$S = B_N \cap V,$$

S would be called a slice of B_N . The direction subspace of S is defined by

$$V_S := \operatorname{span}\{s - s' : s, s' \in S\}.$$

For a collection of slices $\{S_j : j = 1, 2, \dots, p\}$ of B_N , if the dimension of the subspace spanned by the corresponding direction subspaces $\{V_{S_1}, \dots, V_{S_p}\}$ equals to p , then $\{S_j : j = 1, 2, \dots, p\}$ is said to be linear independent.

For any $\varphi \in LFT(B_N)$ and any slice S of B_N , if $\varphi(S) \subset S$, S would be called an invariant slice of φ . Let

$$\# \text{inv}(\varphi) = \dim(\text{span}\{V_S : S \text{ is an invariant slice of } \varphi\}).$$

Then two conclusions are drawn: $\# \text{inv}(\varphi) = 0$ if and only if φ has no invariant slice; $\# \text{inv}(\varphi) = 1$ if and only if φ has only one invariant slice. (For more discussion about slice see [5].)

Definition 2.5 Suppose φ is a self-map of B_N , $z_0 \in B_N$ is a fixed point of φ . $L_U(\varphi, z_0)$ is defined as the unitary space of φ at z_0 if

$$L_U(\varphi, z_0) = \bigoplus_{|\lambda|=1} \ker(d\varphi_{z_0} - \lambda E)^N.$$

The dimension of $L_U(\varphi, z_0)$ is termed the unitary index of φ at z_0 which is denoted by $u(\varphi, z_0)$.

As Lemma 3.1 in [5] has shown, $u(\varphi, z_0) = u(\varphi, z_1)$ for any two fixed point of φ . Thus the unitary index of φ is usually denoted by $u(\varphi)$.

Definition 2.6 For any open subset U of \mathbb{C}^N , the collectivity of holomorphic self-maps of U is denote by $H(U, U)$. $\{\varphi_t : t \geq 0\} \subset H(U, U)$ is a semigroup of holomorphic self-map if

1. $\varphi_0 = id_U$, where $id_U : U \rightarrow U$ is the identity map;
2. $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for ant $s, t \geq 0$;
3. φ_t converges uniformly on compact subsets of U when $t \rightarrow 0^+$.

Throughout this paper, $\{\varphi_t\}$ is called a semigroup for short.

An element of a semigroup $\{\varphi_t\}$ is said to be an iterate of $\{\varphi_t\}$. There are many special properties of semigroup $\{\varphi_t\}$, for instance,

- every iterate of $\{\varphi_t\}$ is an injection;
- if one of the iterates is an automorphism, then all of the iterates are automorphism;
- for any $z \in U$, the map $t \mapsto \varphi_t(z)$ is real analytic.

Semigroups whose iterates are linear fractional maps of B_N have been discussed at [5]. The following is some of the conclusion:

Theorem 2.7 *Suppose $\{\varphi_t\}$ is a semigroup of $H(B_N, B_N)$. If there is a $t_0 \in (0, +\infty)$ such that φ_{t_0} is an elliptic (hyperbolic or parabolic) self-map, then for any $t \in (0, +\infty)$, φ_t is elliptic (hyperbolic or parabolic). Furthermore, if φ_{t_0} is non-elliptic, then all the iterates of $\{\varphi_t\}$ share the same Denjoy-Wolff point.*

According to the above theorem, a semigroup of linear fractional self-maps can be divided into type of elliptic semigroup, type of hyperbolic semigroup and type of parabolic semigroup.

On the other hand, in every semigroup of linear fractional self-maps $\{\varphi_t\}$ of B_N there is a holomorphic map $G : B_N \rightarrow \mathbb{C}^N$ such that for any $t \geq 0$,

$$\frac{\partial \varphi_t}{\partial t} = G \circ \varphi_t.$$

The above map G is said to be the infinitesimal generator of $\{\varphi_t\}$. Besides, one semigroup has only one infinitesimal generator and vice versa (c.f. [6]).

A linear fractional map φ can be embedded into a semigroup composed of linear fractional self-map $\{\varphi_t\}$ if φ is a iterate of $\{\varphi_t\}$. Simple computation shows that if $\varphi : B_N \rightarrow B_N$ is conjugated to $\psi : U \rightarrow U$, which means that there is a biholomorphic map σ between B_N and U , such that

$$\psi = \sigma \circ \varphi \circ \sigma^{-1},$$

φ can be embedded into some semigroup if and only if ψ can be embedded in to some semigroup. Therefore, the study of the embedding problem of semigroup will based on the meaning of conjugation.

3 Normal forms of linear fractional self-maps

3.1 The elliptic cases

Let $\varphi \in LFT(B_N)$ be elliptic. φ has at least one interior fixed point. Therefore, φ is supposed to fix the origin throughout no more than one automorphism. In this case, for any $z \in B_N$,

$$\varphi(z) = \frac{Az}{\langle z, C \rangle + 1},$$

where $A \in \mathbb{C}^{N \times N}$, $C \in \mathbb{C}^N$. By $F = Fix(\varphi)$ is meant the collection of fixed points of φ . According to [12], F is the intersection of B_N and some subspace of \mathbb{C}^N . The dimension of this subspace is denoted by p . Let $u = u(\varphi)$ which is the unitary index of φ at the origin.

Proposition 3.1 *Let $A \in \mathbb{C}^{N \times N}$, and $\|A\| \leq 1$. If λ is an eigenvalue of A with $|\lambda| = 1$, then the geometric multiplicity of λ is equal to 1.*

Proof. According to Schur's Triangularization Theorem (P508, [11]), there is a unitary matrix $U \in \mathbb{C}^{N \times N}$ and an upper-triangular matrix

$$T = \begin{bmatrix} \lambda & a_{12} & \cdots & a_{1N} \\ 0 & \lambda_2 & * & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix},$$

such that

$$A = U^H T U.$$

$\|T\| \leq 1$ for $\|A\| \leq 1$.

Suppose there is a subscript i such that $a_{1i} \neq 0$. Let

$$z_\lambda = \left(\frac{\bar{\lambda}}{\sqrt{1 + |a_{1i}|^2}}, 0, \dots, \frac{\bar{a}_{1i}}{\sqrt{1 + |a_{1i}|^2}}, \dots, 0 \right)^T$$

be a vector of \mathbb{C}^N with

$$|z_\lambda| = 1$$

and

$$T z_\lambda = \left(\frac{\bar{\lambda} \lambda}{\sqrt{1 + |a_{1i}|^2}} + \frac{\bar{a}_{1i} a_{1i}}{\sqrt{1 + |a_{1i}|^2}}, \dots \right)^T = \left(\sqrt{1 + |a_{1i}|^2}, \dots \right).$$

Therefore

$$\|T z_\lambda\| \geq \sqrt{1 + |a_{1i}|^2},$$

and the inequality above contradicts to the fact that $\|T\| \leq 1$. As a consequence, $a_{1i} = 0$ for all $i = 2, \dots, n$. This means that the proposition holds. ■

Proposition 3.2 *Let $\varphi \in LFT(B_N)$ be elliptic with unitary index $u(\varphi) \geq 1$. Then φ is conjugated to $\psi \in LFT(B_N)$ with*

$$\psi(z', z'') = (\Lambda z', A_1 z''),$$

where $(z', z'') \in \mathbb{C}^u \times \mathbb{C}^{N-u} \cap B_N$, Λ is a diagonal and unitary matrix of order u and A_1 is a matrix of order $N - u$ with spectral radius $\rho(A_1) < 1$, $\|A_1\| \leq 1$.

Parts of the following idea come from the proof of Theorem 4.3 in [5].

Proof. Resulting from the previous discussion, we may assume that

$$\varphi(z) = \frac{Az}{\langle z, C \rangle + 1}.$$

Simple computation indicates that the Jacobi matrix of φ at the origin $d\varphi_O = A$. According to Schwartz's lemma (see [12]), $\|A\| \leq 1$. As a consequence of Rudin's version of the Julia–Wolff–Carathéodory theorem (see [12]), there is at least one eigenvalue of A which modules equals 1 since $u \geq 1$. Due to Proposition 3.1, there is a unitary matrix U , such that

$$U^H A U = \begin{bmatrix} \Lambda & \\ & A_1 \end{bmatrix},$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_u \end{bmatrix},$$

and $|\lambda_j| = 1$ for $j = 1, 2, \dots, u$. A_1 is a matrix of order $N - u$ with $\|A_1\| \leq 1$ and $\rho(A_1) < 1$.

Let

$$\begin{aligned} \varphi_1(z', z'') &= U^H (\varphi(Uz)) = U^H \left(\frac{AUz}{\langle Uz, C \rangle + 1} \right) \\ &= \frac{U^H AUz}{\langle z, U^H C \rangle + 1} = \frac{(\Lambda z', A_1 z'')}{\langle z, U^H C \rangle + 1}, \end{aligned}$$

where $(z', z'') \in \mathbb{C}^u \times \mathbb{C}^{N-u} \cap B_N$. Eventually, φ_1 is conjugated to φ . We denote

$$U^H C = (c', c'') \in \mathbb{C}^u \times \mathbb{C}^{N-u},$$

thus

$$\varphi_1(z', z'') = \frac{(\Lambda z', A_1 z'')}{\langle z', c' \rangle + \langle z'', c'' \rangle + 1}.$$

$\varphi_1(\{(z', O) \in \mathbb{C}^u \times \mathbb{C}^{N-u} : |z'| < 1\}) \subset \{(z', O) \in \mathbb{C}^u \times \mathbb{C}^{N-u} : |z'| < 1\}$, consequently

$$\varphi_2(z') \triangleq \varphi_1(z', O) = \frac{\Lambda z'}{\langle z', c' \rangle + 1}$$

is a self-map of the unit ball in \mathbb{C}^u and

$$d(\varphi_2)_O = \Lambda.$$

On the basis of Schwartz's lemma on the ball, φ_2 is linear and as a consequence $c' = 0$. As a result,

$$\varphi_1(z) = \frac{(\Lambda z', A_1 z'')}{\langle z'', c'' \rangle + 1}.$$

We define

$$z_t = \left(\sqrt{1 - t^2 |c''|^2} e'_1, -t c'' \right),$$

then for all $t \in [0, 1]$,

$$|z_t|^2 = \left(\sqrt{1 - t^2 |c''|^2} \right)^2 + t^2 |c''|^2 = 1,$$

and

$$\begin{aligned} \varphi_1(z_t) &= \frac{\left(\sqrt{1 - t^2 |c''|^2} \Lambda e'_1, -t A_1 c'' \right)}{\langle -t c'', c'' \rangle + 1} \\ &= \frac{\left(\sqrt{1 - t^2 |c''|^2} \Lambda e'_1, -t A_1 c'' \right)}{1 - t |c''|^2}. \end{aligned}$$

We obtain

$$\begin{aligned} |\varphi_1(z_t)|^2 &\geq \frac{1 - t^2 |c''|^2}{\left(1 - t |c''|^2 \right)^2}, \\ \frac{d}{dt} \left(\frac{1 - t^2 |c''|^2}{\left(1 - t |c''|^2 \right)^2} \right) &= 2 \frac{|c''|^2}{\left(t |c''|^2 - 1 \right)^3} (t - 1). \end{aligned}$$

Consequently for all $t \in [0, 1)$, $\frac{1 - t^2 |c''|^2}{(1 - t |c''|^2)^2}$ is decreasing for t . Therefore when $t > 0$,

$$|\varphi_1(z_t)|^2 > \lim_{t \rightarrow 0} \frac{1 - t^2 |c''|^2}{\left(1 - t |c''|^2 \right)^2} = 1.$$

It is impossible since $\varphi_1(B_N) \subset B_N$ and φ_1 is holomorphic. As a consequence, $c'' = O$ and

$$\varphi_1(z) = (\Lambda z', A_1 z'').$$

■

Proposition 3.3 *Let φ be a elliptic linear fractional self-map of B_N . Suppose that φ has a unique interior fixed point and unitary index $u(\varphi) = 0$. Then φ is conjugated to $\psi \in LFT(B_N)$ with*

$$\frac{\hat{A}z}{\delta \left\langle z, \left(\hat{A}^H - E \right) e_1 \right\rangle + 1},$$

where \hat{A} is a matrix of order N with $\rho(\hat{A}) < 1$, $\|\hat{A}\| \leq 1$ and $\delta \in [0, 1]$.

Proof. Without loss of generality, we suppose that the unique interior fixed point of φ is the origin. For any $z \in B_N$,

$$\varphi(z) = \frac{Az}{\langle z, C \rangle + 1},$$

where $A \in \mathbb{C}^{N \times N}$, $C \in \mathbb{C}^N$. As a result, $\|A\| \leq 1$. Due to the previous lemma, φ has at least one invariant slice if there is a eigenvalue of A such that $|\lambda| = 1$. This conclusion is not established since $u = 0$. Therefore, $\rho(A) < 1$, both $A - E$ and $A^H - E$ are invertible and there exists a vector $V \in \mathbb{C}^N$ such that

$$C = (A^H - E)V.$$

On the other hand, there is a unitary matrix U such that

$$U^H V = \delta e_1,$$

where $\delta = |V|$, $e_1 = (1, 0, \dots, 0)^T$. Let $\hat{\varphi}(z) = U^H \circ \varphi \circ U(z)$, $\hat{A} = U^H A U$, then

$$\begin{aligned} \hat{\varphi}(z) &= \frac{U^H A U z}{\langle U z, C \rangle + 1} = \frac{U^H A U z}{\langle U z, (A^H - E)V \rangle + 1} \\ &= \frac{U^H A U z}{\langle z, U^H (A^H - E)V \rangle + 1} = \frac{U^H A U z}{\langle z, (U^H A^H U - E)U^H V \rangle + 1} \\ &= \frac{\hat{A}z}{\delta \left\langle z, \left(\hat{A}^H - E \right) e_1 \right\rangle + 1}. \end{aligned}$$

Simple computation shows that

$$\hat{\varphi}_n(z) = \frac{\hat{A}^n z}{\delta \left\langle z, \left(\left(\hat{A}^n \right)^H - E \right) e_1 \right\rangle + 1}.$$

Since $\hat{\varphi}_n(B_N) \subset B_N$ for every $n \in \mathbf{N}$,

$$\left| \delta \left(\left(\hat{A}^n \right)^H - E \right) e_1 \right| \leq 1.$$

$\hat{A}_n \rightarrow O$ when n tends to infinity since A and \hat{A} have the same spectrum. This conclusion leads to $\delta \in [0, 1]$. ■

3.2 Non-elliptic cases

The Siegel half-plane domain of \mathbb{C}^N is defined by

$$\mathbb{H}^N = \left\{ (u_1, u') \in \mathbb{C} \times \mathbb{C}^{N-1} : \text{Im } u_1 > |u'|^2 \right\}.$$

\mathbb{H}^N is biholomorphic with B_N via the Cayley transformation:

$$\sigma(z_1, z') = \left(i \frac{1 + z_1}{1 - z_1}, \frac{iz'}{1 - z_1} \right).$$

Every linear fractional map on B_N is conjugated to a linear fractional map on \mathbb{H}^N since σ is a linear fractional map. Besides, the following lemma can be drawn.

Lemma 3.4 *Suppose $\varphi \in LFT(B_N)$ be non-elliptic with boundary dilation coefficient $\alpha = \alpha(\varphi)$. Then φ is conjugated to a self-map $\tilde{\varphi}$ of \mathbb{H}^N with*

$$\tilde{\varphi}(z, w) = (\lambda z + 2i \langle w, a \rangle + b, Mw + c), \quad (z, w) \in \mathbb{H}^N,$$

where $c \in \mathbb{C}$, $b, d \in \mathbb{C}^{N-1}$, $M \in \mathbb{C}^{(N-1) \times (N-1)}$. Conversely, such a map is a self-map of \mathbb{H}^N if and only if

- (P1) $Q := \lambda I - M^H M$ is a hermitian positive semi-definite matrix;
- (P2) $\text{Im}(b) - |c|^2 \geq \langle Q^+(M^*c - a), M^*c - a \rangle$ where Q^+ is the pseudo-inverse of Q (for more details about pseudo-inverse, we refer to [11], p422);
- (P3) $QQ^+(M^*c - a) = M^*c - a$.

Proof. This lemma is a modified version of theorem 4.1 of [5]. The only difference is (P3). In fact, the corresponding condition is $M^*c - a$ belongs to the space spanned by the columns of Q in Theorem 4.1 of [5]. That is to say, there is a vector $x \in \mathbb{C}^{N-1}$ such that

$$Qx = M^*c - a.$$

According to [11], this equation has at least one solution if and only if

$$QQ^+(M^*c - a) = M^*c - a$$

holds. ■

The following lemma about normal forms of parabolic linear fractional self-maps results from section 2 of [1].

Lemma 3.5 *Let $\varphi \in LFM(B_N)$ be parabolic. Then φ is conjugated to $\psi \in LFM(\mathbb{H}^N)$ with*

$$\psi(z, u, v, w) = (z + 2i\langle u, a \rangle + 2i\langle w, c \rangle + b, u + a, Dv, Aw), \quad (1)$$

where

1. D is diagonal, $\sigma(D) \subset \partial\mathbb{D} \setminus \{1\}$;
2. $Q = I - A^H A$ is a hermitian positive semi-definite matrix;
3. $\text{Im}(b) - |a|^2 \geq \langle Q^+c, c \rangle$;
4. $QQ^+c = c$.

Sometimes there are no u or no v or no w appeared in (1). Further more, due to theorem 4.4 of [5], we may assume that $a = 0$ if φ has at least one invariant slice.

Theorem 3.6 *Let $\varphi \in LFT(B_N)$ be hyperbolic. Then φ is conjugated to $\psi \in LFT(\mathbb{H}^N)$ with*

$$\psi(z, u, v, w) = (\lambda z + b, \sqrt{\lambda}u, \sqrt{\lambda}Dv, \sqrt{\lambda}Aw + c),$$

where

1. D is diagonal, $\sigma(D) \subset T \setminus \{1\}$;
2. Both $Q = I - A^H A$ and $P = I - AA^H$ are hermitian positive semi-definite matrix;

3. $\text{Im}(b) \geq \langle P^+c, c \rangle$ where P^+ is the pseudo-inverse of Q ;

4. $QQ^+A^Hc = A^Hc$.

Proof. Let $\varphi \in LFM(B_N, B_N)$ be hyperbolic. According to theorem 4.1 in [5], φ is conjugated to a self-map $\tilde{\varphi}$ of \mathbb{H}^N with

$$\tilde{\varphi}(z, W) = (\lambda z + 2i \langle W, a \rangle + b, MW + c), \quad (z, W) \in \mathbb{H}^N,$$

where $c \in \mathbb{C}$, $b, d \in \mathbb{C}^{N-1}$, $A \in \mathbb{C}^{(N-1) \times (N-1)}$ satisfy the following conditions:

- (1) $Q_M := \lambda I - M^H M$ is a Hermit semi-definite matrix;
- (2) $\text{Im}(b) - |c|^2 \geq \langle Q_M^+(M^*c - a), M^*c - a \rangle$ where Q^+ is the pseudo-inverse of Q ,
- (3) $Q_M Q_M^+(M^*c - a) = M^*c - a$.

We may assume that

$$\frac{1}{\sqrt{\lambda}}M = \begin{pmatrix} I & & \\ & D & \\ & & A \end{pmatrix}$$

since

$$\left\| \left(\frac{1}{\sqrt{\lambda}}M \right) \left(\frac{1}{\sqrt{\lambda}}M \right)^H \right\| \leq 1,$$

where D is diagonal with $\sigma(D) \subset \partial\mathbb{D} \setminus \{1\}$ and $\rho(A) < 1$, $\|A\| \leq 1$. That is why $\tilde{\varphi}$ could be written as the following form:

$$\tilde{\varphi}(z, W) = (\lambda z + 2i \langle u, a_1 \rangle + 2i \langle v, a_2 \rangle + 2i \langle w, a_3 \rangle + b, \sqrt{\lambda}u + c_1, \sqrt{\lambda}Dv + c_2, \sqrt{\lambda}Aw + c_3).$$

Let τ be an automorphism of \mathbb{H}^N which preserves ∞ with

$$\tau(z, W) = (z + 2i \langle W, \gamma \rangle + \beta, W + \gamma)$$

and

$$\tau^{-1}(z, W) = \left(z - 2i \langle W, \gamma \rangle - \beta + 2i |\gamma|^2, W - \gamma \right).$$

We set $\psi_1 = \tau \circ \tilde{\varphi} \circ \tau^{-1}$, then

$$\begin{aligned} \psi_1(z, W) = & (\lambda z + 2i \langle W, \tilde{a} \rangle + \tilde{b}, \sqrt{\lambda}u + (1 - \sqrt{\lambda})\gamma_1 + c_1, \\ & \sqrt{\lambda}Dv + (I - \sqrt{\lambda}D)\gamma_2 + c_2, \sqrt{\lambda}Aw + (I - \sqrt{\lambda}A)\gamma_3 + c_3) \end{aligned}$$

for some \tilde{a}, \tilde{b} . Choose γ_1 and γ_2 such that $(1 - \sqrt{\lambda})\gamma_1 + c_1 = 0$ and $(I - \sqrt{\lambda}D)\gamma_2 + c_2 = 0$, we can obtain

$$\psi_1(z, W) = \left(\lambda z + 2i \langle u, \tilde{a}_1 \rangle + 2i \langle v, \tilde{a}_2 \rangle + 2i \langle w, \tilde{a}_3 \rangle + \tilde{b}, \sqrt{\lambda}u, \sqrt{\lambda}Dv, \sqrt{\lambda}Aw + \tilde{c} \right).$$

Let

$$\tilde{Q} = \lambda I - \sqrt{\lambda} \begin{pmatrix} I & & \\ & D & \\ & & A \end{pmatrix} \cdot \sqrt{\lambda} \begin{pmatrix} I & & \\ & D & \\ & & A \end{pmatrix}^H = \lambda \begin{pmatrix} 0 & & \\ & 0 & \\ & & I - A^H A \end{pmatrix}.$$

\tilde{Q} is hermitian semi-definite positive matrix since ψ_1 is a self-map of \mathbb{H}^N fixing ∞ . As a result, $I - A^H A$ is hermitian semi-definite positive. We may get $\tilde{a}_1 = 0, \tilde{a}_2 = 0$ based on (3) and

$$\tilde{Q}^+ = \frac{1}{\lambda} \begin{pmatrix} 0 & & \\ & 0 & \\ & & (I - A^H A)^+ \end{pmatrix},$$

Let τ_1 be an automorphism of \mathbb{H}^N with

$$\begin{aligned} \tau_1(z, u, v, w) &= (z + 2i \langle w, \gamma \rangle + \beta_1, u, v, w + \gamma), \\ \tau_1^{-1}(z, u, v, w) &= (z - 2i \langle w, \gamma \rangle - \beta_1 + 2i |\gamma|^2, u, v, w - \gamma). \end{aligned}$$

Let $\psi = \tau \circ \psi_1 \circ \tau^{-1}$, thus

$$\begin{aligned} \psi(z, u, v, w) &= \tau \circ \psi_1 \left(z - 2i \langle w, \gamma \rangle - \beta_1 + 2i |\gamma|^2, u, v, w - \gamma \right) \\ &= \tau \left(\lambda \left(z - 2i \langle w, \gamma \rangle - \beta_1 + 2i |\gamma|^2 \right) + 2i \langle w - \gamma, \tilde{a}_3 \rangle + \tilde{b}, \sqrt{\lambda}u, \sqrt{\lambda}Dv, \sqrt{\lambda}A(w - \gamma) + \tilde{c} \right) \\ &= \left(\lambda z + 2i \langle w, \tilde{a}_3 + \sqrt{\lambda}A^H \gamma - \lambda \gamma \rangle + \tilde{b}_1, \sqrt{\lambda}u, \sqrt{\lambda}Dv, \sqrt{\lambda}A + \tilde{c}_1 \right). \end{aligned}$$

We may choose γ such that $\tilde{a}_3 + \sqrt{\lambda}A^H \gamma - \lambda \gamma = 0$ since $\rho(A) < 1$. If we write \tilde{b}_1 as b and \tilde{c}_1 as c again, the result will be

$$\psi(z, u, v, w) = \left(\lambda z + b, \sqrt{\lambda}u, \sqrt{\lambda}Dv, \sqrt{\lambda}A + c \right).$$

According to (1), (2) and (3), we obtain

- Both $Q = I - A^H A$ and $P = I - A A^H$ are hermitian positive semi-definite matrix;

- $\text{Im } b - |c|^2 \geq \langle Q^+ A^H c, A^H c \rangle$;
- $QQ^+ A^H c = A^H c$.

Let $A = U\Sigma V$ be the singular value decomposition of A , namely U and V are unitary, Σ is diagonal and the coefficients on the diagonal of Σ are non-negative. Let

$$\Sigma = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix},$$

where $1 \notin \sigma(B)$. According to this decomposition,

$$\begin{aligned} Q &= V^H \begin{bmatrix} 0 & 0 \\ 0 & I - B^2 \end{bmatrix} V, Q^+ = V^H \begin{bmatrix} 0 & 0 \\ 0 & (I - B^2)^{-1} \end{bmatrix} V, \\ P &= U \begin{bmatrix} 0 & 0 \\ 0 & I - B^2 \end{bmatrix} U^H, P^+ = U \begin{bmatrix} 0 & 0 \\ 0 & (I - B^2)^{-1} \end{bmatrix} U^H, \end{aligned}$$

and

$$QQ^+ A^H c = V^H \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} U^H c.$$

Therefore, $QQ^+ A^H c = A^H c$ is equivalent to

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^H c = 0.$$

Consequently,

$$\begin{aligned} \langle Q^+ A^H c, A^H c \rangle + |c|^2 &= c^H U \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} V V^H \begin{bmatrix} 0 & 0 \\ 0 & (I - B^2)^{-1} \end{bmatrix} V V^H \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} U^H c + |c|^2 \\ &= c^H U \begin{bmatrix} 0 & 0 \\ 0 & B^2 (I - B^2)^{-1} \end{bmatrix} U^H c + c^H U \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} U^H c \\ &= c^H U \begin{bmatrix} 0 & 0 \\ 0 & (I - B^2)^{-1} \end{bmatrix} U^H c \\ &= \langle P^+ c, c \rangle. \end{aligned}$$

■

The proof of the theorem above actually has proved the following corollary:

Corollary 3.7 *Let $\varphi \in LFT(B_N)$ be hyperbolic. Then φ is conjugated to $\psi \in LFT(\mathbb{H}^N)$ with*

$$\psi(z, u, v, w) = \left(\lambda z + 2i \langle w, a \rangle + b, \sqrt{\lambda} u, \sqrt{\lambda} Dv, \sqrt{\lambda} Aw + c \right),$$

where

1. D is diagonal, $\sigma(D) \subset \partial\mathbb{D} \setminus \{1\}$;
2. $Q = I - A^H A$ is hermitian positive semi-definite matrix;
3. $\text{Im}(b) - |c|^2 \geq \left\langle Q^+ \left(A^H c - \frac{a}{\sqrt{\lambda}} \right), A^H c - \frac{a}{\sqrt{\lambda}} \right\rangle$ where Q^+ is the pseudo-inverse of Q ;
4. $QQ^+ \left(A^H c - \frac{a}{\sqrt{\lambda}} \right) = A^H c - \frac{a}{\sqrt{\lambda}}$.

In the case of parabolic automorphism, the following two corollaries are formed according to Proposition 4.3 in [5] and Lemma 3.5.

Corollary 3.8 *Let φ be a parabolic automorphism of B_N . Then φ is conjugated to $\psi \in \text{Aut}(\mathbb{H}^N, \mathbb{H}^N)$ with*

$$\psi(z, u, v) = \left(z + 2i \langle u, a \rangle + i |a|^2 + b, u + a, Dv \right)$$

where b is a real number and D is diagonal, $\sigma(D) \subset \partial\mathbb{D} \setminus \{1\}$.

Corollary 3.9 *Let φ be a hyperbolic automorphism of B_N , then φ is conjugated to $\psi \in \text{Aut}(\mathbb{H}^N, \mathbb{H}^N)$ with*

$$\psi(z, W) = \left(\lambda z + b, \sqrt{\lambda} U W \right)$$

where $(z, W) \in \mathbb{C} \times \mathbb{C}^{N-1}$, b is a real number and U is a unitary matrix.

4 Embedding problems

4.1 The elliptic case

Owing to the previous discussions, we would consider the embedding problem of the normal forms only.

Theorem 4.1 *Let $\varphi \in LFT(B_N)$ be elliptic with*

$$\varphi(z', z'') = (\Lambda z', Az'')$$

where Λ is a diagonal unitary matrix and $\rho(A) < 1$, $\|A\| \leq 1$. φ can be embedded into a semigroup of linear fractional maps of B_N if and only if there is a dissipative matrix M and $\sigma(M)$ is a subset of the left half plane such that

$$\exp(M) = A.$$

Proof. We can find a real diagonal matrix Θ such that Λ can be written as

$$\Lambda = \exp(i\Theta),$$

for Λ is a diagonal unitary matrix.

Suppose firstly that there is a dissipative matrix M such that $\exp(M) = A$. Hence,

$$\varphi(z', z'') = (\exp(i\Theta) z', \exp(M) z'').$$

Let

$$\varphi_t(z', z'', z''') = (\exp(it\Theta) z', \exp(tM) z'').$$

$\varphi_t(B_N) \subset B_N$ since $\|\exp(it\Theta)\| = 1$ and $\|\exp(tM)\| \leq 1$. Moreover, $\varphi_{t+s} = \varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t$. As a result, $\{\varphi_t\}$ is a semigroup of B_N with $\varphi_1 = \varphi$.

On the other hand, if φ can be embedded into a semigroup of linear fractional self-maps $\{\varphi_t\}$, φ_t is conjugated to the following linear fractional map owing to Theorem 3.2 of [5]:

$$\psi_t(z', z'') = (\exp(it\tilde{\Theta}) z', \exp(t\tilde{M}) z''),$$

where $\tilde{\Theta}$ is a real diagonal matrix, \tilde{M} is dissipative and all eigenvalues locates on the left half plane. Suppose that the corresponding index of φ is t_0 . Thus φ is conjugated to

$$\psi_{t_0}(z', z'') = (\exp(it_0\tilde{\Theta}) z', \exp(t_0\tilde{M}) z'').$$

There are unitary matrixes U and V such that

$$\Lambda = U^H \exp(it_0\tilde{\Theta}) U \text{ and } A = V^H \exp(t_0\tilde{M}) V = \exp(t_0 V^H \tilde{M} V),$$

for every automorphism of B_N which fixes the origin is a unitary transformation. Let $M = t_0 V^H \tilde{M} V$. Therefore,

$$A = \exp(M)$$

where M is dissipative and all eigenvalues locate on the left half plane. ■

We will make use of the following generalization of Berkson-Porta's criterion due to Aharonov, Elin, Reich and Shoikhet (see Theorem 1.3, [1]):

Lemma 4.2 *Let $F : B_N \rightarrow \mathbb{C}^N$ be holomorphic. F is the infinitesimal generator of a semigroup of holomorphic self-maps of B_N fixing the origin if and only if*

$$F(z) = -Q(z)z$$

where $Q(z)$ is a matrix of order N with holomorphic entries such that

$$\operatorname{Re} \langle Q(z), z \rangle \geq 0.$$

Due to the above discussion, we can prove the following theorem.

Theorem 4.3 *Let $\varphi \in LFT(B_N)$ with unique interior fixed point and*

$$\varphi(z) = \frac{Az}{\delta \langle z, (A^H - E)e_1 \rangle + 1},$$

where $\delta \in [0, 1]$, $e_1 = (1, 0, \dots, 0)^T$. φ can be embedded into a semigroup of linear fractional self-map of B_N if and only if there is a matrix of order N such that $A = \exp(M)$, and

$$\operatorname{Re} \left[\langle Mz, z \rangle - \delta \langle Mz, e_1 \rangle |z|^2 \right] \geq 0.$$

Proof. Due to [12], A and φ share the same fixed point. As a consequence, 1 is not an eigenvalue of A since φ has only one interior fixed point.

If φ can be embedded into $\{\varphi_t\}$ which is a semigroup of linear fractional self-maps of B_N . Without loss of generality, we may assume that $\varphi = \varphi_1$. Due to Theorem 3.2 of [5], there is a matrix M such that

$$\varphi_t(z) = \frac{\exp(tM)z}{\delta \left\langle z, \left(\exp(tM)^H - E \right) e_1 \right\rangle + 1}.$$

$A = \exp(M)$ for $\varphi = \varphi_1$. Easy computation gives

$$\frac{d}{dt} \varphi_t(z) = (M - \delta \langle M \varphi_t(z), e_1 \rangle E) \varphi_t(z).$$

Thus

$$F(z) = (M - \delta \langle Mz, e_1 \rangle E)z$$

is the infinitesimal generator of $\{\varphi_t\}$. By Lemma 4.2, we obtain

$$\operatorname{Re} \langle (M - \delta \langle Mz, e_1 \rangle E) z, z \rangle = \operatorname{Re} \left[\langle Mz, z \rangle - \delta \langle Mz, e_1 \rangle \|z\|^2 \right] \geq 0.$$

On the other hand, if there is a matrix M such that $A = \exp(M)$, and

$$\operatorname{Re} \left[\langle Mz, z \rangle - \delta \langle Mz, e_1 \rangle |z|^2 \right] \geq 0.$$

then $F(z) = (M - \delta \langle Mz, e_1 \rangle E) z$ is the infinitesimal generator of some semigroup. Denote this semigroup by $\{\varphi_t\}$, then

$$\frac{d}{dt} \varphi_t(z) = F(\varphi_t(z)).$$

Hence we have the following differential equalities

$$\begin{cases} \frac{d}{dt} \varphi_t(z) = F(\varphi_t(z)) \\ \varphi_0(z) = z \end{cases},$$

such an initial problem has unique solution according to the theory of differential equation. Due to the previous discussion,

$$\varphi_t(z) = \frac{\exp(tM) z}{\delta \left\langle z, \left(\exp(tM)^H - E \right) e_1 \right\rangle + 1}$$

is just the solution of this equation. Therefore $\{\varphi_t\}$ is a semigroup of linear fractional self-maps of B_N with

$$\varphi_1(z) = \varphi(z).$$

■

4.2 The parabolic cases

The embedded problem of non-elliptic case is much more complicated than the cases of elliptic ones. Some known conclusion can be found in [5].

For any $\alpha, \beta \in \mathbb{C}$, $a \in \mathbb{C}^p$, $D \in \mathbb{C}^{q \times q}$, $A \in \mathbb{C}^{r \times r}$, let

$$\begin{aligned} \tau(z, W) &= (z + 2i \langle u, a \rangle + \beta, u + a, Dv, w), \\ \rho(z, W) &= (z + 2i \langle w, c \rangle + \alpha, u, v, Aw). \end{aligned}$$

Easy computation shows that

$$\tau \circ \rho = \rho \circ \tau. \tag{2}$$

Theorem 4.4 *Let $\psi \in LFT(\mathbb{H}^N)$ be parabolic with*

$$\psi(z, u, v, w) = (z + 2i\langle w, c \rangle + b, u, v, Aw),$$

where $\rho(A) < 1$ and $\|A\| \leq 1$. Let

$$\exp(M) = A,$$

and

$$\begin{aligned} c_t &= \left(I - \exp(M)^H\right)^{-1} \left(I - \exp(tM)^H\right) c, \\ Q_t &= I - \exp(tM)^H \exp(tM), \\ b_t &= tb. \end{aligned}$$

If

1. *M is dissipative;*
2. *for any $t \geq 0$, $t \operatorname{Im} b \geq \lambda^{-t} \langle Q_t^+ c_t, c_t \rangle$,*
3. *$Q_t Q_t^+ c_t = c_t$,*

then ψ can be embedded into a semigroup of \mathbb{H}^N .

Proof. Let

$$\psi_t(z, u, v, w) = (u + 2i\langle w, c_t \rangle + b_t, u, v, \exp(tM)w).$$

ψ_t is a self-map of \mathbb{H}^N for every $t \geq 0$ according to Theorem 3.4 for any $t \geq 0$. When $t = 0$, we have

$$c_0 = 0, b_0 = 0, \exp(0M) = E.$$

Thereby,

$$\psi_0(z, u, v, w) = (z, u, v, w).$$

Direct computation shows that for any $s, t \geq 0$,

$$\psi_s \circ \psi_t = \psi_t \circ \psi_s = \psi_{s+t}.$$

That ψ_t converges uniformly on compact subset of \mathbb{H}^N is clear. As a consequence, $\{\psi_t\}$ is a semigroup of \mathbb{H}^N . Due to $\psi_1 = \psi$, ψ can be embedded into a semigroup of \mathbb{H}^N . ■

We will consider the case that the matrix B showed up at the previous theorem is normal. Up to a conjugation with an automorphism, we may suppose that B is diagonal.

Lemma 4.5 *Let*

$$h(u, v, t) = \frac{1 + \exp(-2tu) - 2 \exp(-tu) \cos(vt)}{(1 - \exp(-2tu))t}.$$

Then

$$h(u, v, t) \leq \lim_{t \rightarrow 0^+} h(u, v, t) = \frac{1}{2u} (u^2 + v^2)$$

for any $u > 0, v \geq 0, t \geq 0$.

Proof. Rewrite $h(u, v, t)$ by

$$\begin{aligned} h(u, v, t) &= \frac{(1 - \exp(-tu))}{(1 + \exp(-tu))t} + \frac{2 \exp(-tu) (1 - \cos(vt))}{(1 - \exp(-2tu))t} \\ &\triangleq h_1(u, t) + h_2(u, v, t). \end{aligned}$$

We obtain

$$\frac{\partial h_1}{\partial t} = \frac{1}{t^2 (e^{-tu} + 1)^2} (e^{2(-tu)} + 2tue^{-tu} - 1).$$

Let

$$r(x) = \exp(-2x) + 2x \exp(-x) - 1.$$

Then

$$r'(x) = -2e^{-x} (x + e^{-x} - 1).$$

Denote by x_0 any one of the solutions of $r'(x) = 0$, then

$$e^{-x_0} = 1 - x_0,$$

and

$$r(x_0) = (1 - x_0)^2 + 2x_0(1 - x_0) - 1 = -x_0^2 \leq 0,$$

For any $x \in (0, +\infty)$, $r(x) \leq 0$ since $r(0) = 0$, $\lim_{x \rightarrow \infty} r(x) = -1$ and $r(x_0) \leq 0$. As a consequence, $h_1(u, t)$ is decreasing when u fixed. Therefore,

$$h_1(u, t) \leq \lim_{t \rightarrow 0^+} h_1(u, t).$$

Rewrite h_2 by

$$\begin{aligned} h_2(u, v, t) &= \frac{2t \exp(-tu)}{(1 - \exp(-2tu))} \cdot \frac{(1 - \cos(vt))}{t^2} \\ &= \frac{v^2}{u} \frac{2tu \exp(-tu)}{(1 - \exp(-2tu))} \cdot \frac{(1 - \cos(vt))}{(vt)^2}. \end{aligned}$$

One has

$$\frac{d}{dx} \left(\frac{2x \exp(-x)}{1 - \exp(-2x)} \right) = -2 \frac{e^{-3x}}{(e^{2(-x)} - 1)^2} (xe^{2x} + 1 + x - e^{2x}),$$

and

$$\frac{d}{dx} (xe^{2x} + 1 + x - e^{2x}) = 2xe^{2x} - e^{2x} + 1.$$

For all $x \geq 0$, $2xe^{2x} - e^{2x} + 1 \geq 0$ because $2xe^{2x} - e^{2x} + 1 = 0$ has only one solution $x = 0$ and

$$\lim_{x \rightarrow +\infty} (2xe^{2x} - e^{2x} + 1) = +\infty.$$

As a result, for any $x \in (0, +\infty)$,

$$\frac{d}{dx} \left(\frac{2x \exp(-x)}{1 - \exp(-2x)} \right) \leq 0.$$

Consequently,

$$\frac{2x \exp(-x)}{1 - \exp(-2x)} \leq \lim_{x \rightarrow 0} \frac{2x \exp(-x)}{1 - \exp(-2x)} = 1.$$

Moreover,

$$\frac{1 - \cos x}{x^2} = \frac{2 \sin^2 \frac{x}{2}}{x^2} \leq \frac{2 \left(\frac{x}{2}\right)^2}{x^2} = \frac{1}{2}.$$

Thus for all $t > 0$,

$$\frac{2tu \exp(-tu)}{(1 - \exp(-2tu))} \leq \lim_{t \rightarrow 0} \frac{2tu \exp(-tu)}{(1 - \exp(-2tu))} = 1,$$

and

$$\frac{(1 - \cos(vt))}{(vt)^2} \leq \lim_{t \rightarrow 0} \frac{(1 - \cos(vt))}{(vt)^2} = \frac{1}{2}.$$

As a consequence, for any $t \geq 0$,

$$h_2(u, v, t) \leq h_2(u, v, 0).$$

Therefore, when $t \geq 0$, we obtain

$$h(u, v, t) = \frac{|1 - \exp((-u + iv)t)|^2}{(1 - \exp(-2ut))t} \leq \lim_{t \rightarrow 0} h(u, v, t) = \frac{1}{2u} (u^2 + v^2).$$

■

Corollary 4.6 *Let $\psi \in LFT(\mathbb{H}^N)$ be parabolic with*

$$\psi(z, u, v, w) = (z + 2i \langle w, c \rangle + b, u, v, Aw)$$

where $A = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r)$ and $|\lambda_j| < 1$ for $j = 1, 2, \dots, r$. Let $\lambda_j = \exp(-u_j + iv_j)$ where $u_j > 0, v_j \geq 0$ for $j = 1, 2, \dots, r$. If

$$\text{Im } b \geq c^H \Theta c$$

where

$$\Theta = \text{diag} \left(\frac{1}{2u_1} \frac{(u_1^2 + v_1^2)}{|1 - \lambda_1|^2}, \dots, \frac{1}{2u_r} \frac{(u_r^2 + v_r^2)}{|1 - \lambda_r|^2} \right),$$

then ψ can be embedded into a semigroup of \mathbb{H}^N .

Proof. Let

$$M = \text{diag}(-u_1 + iv_1, \dots, -u_r + iv_r).$$

Then

$$A = \exp(M).$$

Denote by

$$\begin{aligned} c_t &= (I - \exp(M^H))^{-1} (I - \exp(tM^H)) c, \\ Q_t &= I - (\exp(tM))^H \exp(tM), \\ b_t &= tb. \end{aligned}$$

Since both A and M are normal matrix and $\|B\| = \|\exp(M)\|$,

$$\|\exp(M)\| \leq 1.$$

According to proposition 2.4, for any $t \geq 0$,

$$\|\exp(tM)\| \leq 1.$$

Therefore Q_t is hermitian positive semi-definite and

$$Q_t^+ = Q_t^{-1} = \begin{bmatrix} \frac{1}{1-e^{-2tu_1}} & & & \\ & \frac{1}{1-e^{-2tu_2}} & & \\ & & \ddots & \\ & & & \frac{1}{1-e^{-2tu_r}} \end{bmatrix}.$$

Besides,

$$c_t = (E - \exp(M^H))^{-1} (E - \exp(tM^H)) c$$

$$= \begin{bmatrix} \frac{1-e^{t(-u_1-iv_1)t}}{1-e^{(-u_1-iv_1)t}} & & & \\ & \frac{1-e^{t(-u_2-iv_2)t}}{1-e^{(-u_2-iv_2)t}} & & \\ & & \ddots & \\ & & & \frac{1-e^{t(-u_r-iv_r)t}}{1-e^{(-u_r-iv_r)t}} \end{bmatrix} c.$$

As a result,

$$c_t^H Q_t^+ c_t = c^H \Theta_t c,$$

where

$$\Theta_t = \text{diag} \left(\frac{|1-e^{t(-u_1+iv_1)}|^2}{|1-\lambda_1|^2(1-e^{-2tu_1})}, \dots, \frac{|1-e^{t(-u_r+iv_r)}|^2}{|1-\lambda_r|^2(1-e^{-2tu_r})} \right).$$

Denote by

$$b = (\beta_1, \beta_2, \dots, \beta_p)^T,$$

then

$$b_t^H Q_t^+ b_t = \sum_{j=1}^p \frac{|1-e^{t(-u_j+iv_j)}|^2}{|1-\lambda_j|^2(1-e^{-2tu_j})} |\beta_j|^2.$$

Let

$$g_{\lambda_j}(t) = \frac{1}{t} \frac{|1-e^{t(-u_j+iv_j)}|^2}{|1-\lambda_j|^2(1-e^{-2tu_j})}.$$

According to Lemma 4.5, for $j = 1, 2, \dots, r$,

$$\sup_{t \geq 0} g_{\lambda_j}(t) = \frac{1}{2u_j} (u_j^2 + v_j^2) \frac{1}{|1-\lambda_j|^2}.$$

Further more,

$$\begin{aligned} \sup_{t \geq 0} \left\{ \frac{1}{t} c_t^H Q_t^+ c_t \right\} &= \sum_{j=1}^p \frac{1}{2u_j} (u_j^2 + v_j^2) \frac{1}{|1-\lambda_j|^2} |\beta_j|^2 \\ &= c^H \Theta c. \end{aligned}$$

Consequently

$$c_t^H Q_t^+ c_t \leq t \text{Re } c.$$

Due to Theorem 4.4, ψ can be embedded into a semigroup of \mathbb{H}^N . ■

Lemma 4.7 *Let ψ be a parabolic automorphism of \mathbb{H}^N with*

$$\psi(z, u, v) = \left(z + 2i \langle u, a \rangle + i |a|^2 + b, u + a, Dv \right)$$

where b is a real number and D is diagonal, $\sigma(D) \subset T \setminus \{1\}$. Then φ can be embedded into a semigroup of B_N .

Proof. Let

$$\psi_t(z, u, v) = \left(z + 2i \langle u, ta \rangle + it^2 |a|^2 + tb, u + ta, \exp(it\Theta) v'' \right),$$

then clearly, for every $t > 0$, ψ_t is an automorphism of \mathbb{H}^N and $\{\psi_t\}$ is a semigroup of \mathbb{H}^N . Therefore ψ can be embedded into some semigroup of B_N . ■

Theorem 4.8 *Let $\psi \in LFT(\mathbb{H}^N)$ be parabolic with*

$$\psi(z, u, v, w) = (z + 2i \langle u, a \rangle + 2i \langle w, c \rangle + b, u + a, Dv, Aw)$$

where $A = \text{diag}(\lambda_1, \dots, \lambda_r)$ and $|\lambda_j| < 1$ for $j = 1, 2, \dots, r$. Let $\lambda_j = \exp(-u_j + iv_j)$ where $u_j > 0, v_j \in [0, 2\pi)$ for $j = 1, 2, \dots, r$. If

$$\text{Im } b - |a|^2 \geq c^H \Theta c$$

where

$$\Theta = \text{diag} \left(\frac{1}{2u_1} \frac{(u_1^2 + v_1^2)}{|1 - \lambda_1|^2}, \dots, \frac{1}{2u_r} \frac{(u_r^2 + v_r^2)}{|1 - \lambda_r|^2} \right),$$

then ψ can be embedded into a semigroup of \mathbb{H}^N .

Proof. Let

$$\begin{aligned} \tau_\psi(z, W) &= (z + 2i \langle u, a \rangle + \beta_\psi, u + a, Dv, w), \\ \rho_\psi(z, W) &= (z + 2i \langle w, c \rangle + \alpha_\psi, u, v, Aw), \end{aligned}$$

with $b = \alpha_\psi + \beta_\psi$ and $\text{Im}(\beta_\psi) = |a|^2, \alpha_\psi \in \mathbb{R}$. Then τ_ψ is a parabolic automorphism and according to Lemma 4.7, τ_ψ can be embedded into $\tau_{\psi,t}$ with

$$\tau_{\psi,t}(z, u, v, w) = \left(z + 2i \langle u, ta \rangle + it^2 |a|^2 + t \text{Re } \beta_\psi, u + ta, \exp(t\Theta_D) v, w \right).$$

According to Corollary 4.6, ρ_ψ can be embedded into a semigroup $\rho_{\psi,t}$ of \mathbb{H}^N with

$$\rho_{\psi,t}(z, u, v, w) = (z + 2i \langle w, c_t \rangle + \alpha_{\psi,t}, u, v, \exp(t\Theta_A) w)$$

for some suitable $c_t, \alpha_{\psi,t}$ and $\exp(t\Theta_A)$. Let

$$\psi_t = \tau_{\psi,t} \circ \rho_{\psi,t},$$

then

$$\begin{aligned} \psi_{t+s} &= \tau_{\psi,t+s} \circ \rho_{\psi,t+s} \\ &= \tau_{\psi,t} \circ \tau_{\psi,s} \circ \rho_{\psi,s} \circ \rho_{\psi,t} \\ &= \tau_{\psi,s} \circ \tau_{\psi,t} \circ \rho_{\psi,s} \circ \rho_{\psi,t} \\ &= \tau_{\psi,s} \circ \rho_{\psi,s} \circ \tau_{\psi,t} \circ \rho_{\psi,t} = \psi_s \circ \psi_t \\ &= \psi_t \circ \psi_s. \end{aligned}$$

As a consequence, ψ can be embedded into a semigroup of \mathbb{H}^N . ■

4.3 The hyperbolic case

Just as Theorem 4.4, in the case of hyperbolic, we have:

Proposition 4.9 *Let $\psi \in LFT(\mathbb{H}^N)$ with*

$$\psi(z, u, v, w) = \left(\lambda z + 2i \langle w, a \rangle + b, \sqrt{\lambda}u, \sqrt{\lambda}Dv, \sqrt{\lambda}Aw \right),$$

where $\lambda > 1$, D is diagonal and $\sigma(D) \subset T \setminus \{1\}$. Suppose A is non-singular and there exists a matrix M such that

$$\exp(M) = B.$$

Denoted by

$$\begin{aligned} \lambda_t &= \lambda^t, \\ A_t &= \exp(tM), \\ a_t &= \left(\lambda - \sqrt{\lambda}A^H \right)^{-1} \left(\lambda_t - \sqrt{\lambda_t}A_t^H \right) a, \\ b_t &= \frac{1 - \lambda_t}{1 - \lambda}, \\ Q_t &= E - A_t^H A_t. \end{aligned}$$

If

1. Q_t is Hermitian positive semi-definite;

2. for any $t \geq 0$,

$$\operatorname{Im} b_t \geq \frac{1}{\lambda_t} \langle Q_t^+ a_t, a_t \rangle;$$

3. for any $t \geq 0$, $Q_t Q_t^+ a_t = a_t$,

then ψ can be embedded in to a semigroup of \mathbb{H}^N .

The proof of the above theorem is just the same with Theorem 4.4, we omit it here.

Lemma 4.10 *Let*

$$h(\lambda, u, v, t) = \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{|e^{t(u+iv)} - 1|^2}{(1 - e^{\lambda t} e^{2ut})},$$

Then

$$h(\lambda, u, v, t) \leq \lim_{t \rightarrow 0^+} h(\lambda, u, v, t) = -\frac{1}{\lambda(2u + \lambda)} (u^2 + v^2)$$

for any $\lambda + 2u < 0, \lambda > 0, v \geq 0$ and $u < 0$.

Proof. Since

$$\begin{aligned} h(\lambda, u, v, t) &= \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{(1 - \exp(ut))^2 + 2 \exp(ut) (1 - \cos vt)}{(1 - e^{\lambda t} e^{2ut})} \\ &= \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{(1 - \exp(ut))^2}{(1 - e^{\lambda t} e^{2ut})} + \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{2 \exp(ut) (1 - \cos vt)}{(1 - e^{\lambda t} e^{2ut})}, \end{aligned}$$

let

$$h_1(u, \lambda, t) = \ln \left(\frac{1}{(1 - e^{-\lambda t})} \cdot \frac{(1 - \exp(ut))^2}{(1 - e^{\lambda t} e^{2ut})} \right),$$

then

$$\begin{aligned} \frac{d}{dt} h_1(u, \lambda, t) &= \frac{d}{dt} \ln \left(\frac{1}{(1 - e^{-\lambda t})} \cdot \frac{(1 - \exp(ut))^2}{(1 - e^{\lambda t} e^{2ut})} \right) \\ &= \frac{1}{t} \left(-\frac{2ut \exp(ut)}{1 - \exp(ut)} + \frac{-\lambda t \exp(-\lambda t)}{1 - \exp(-\lambda t)} + \frac{(\lambda + 2u) t \exp(\lambda + 2u) t}{1 - \exp(\lambda + 2u) t} \right). \end{aligned}$$

Denote

$$g(x) = \frac{x \exp(x)}{1 - \exp(x)},$$

then

$$g''(x) = \frac{d^2}{dx^2} \left(\frac{x \exp(x)}{1 - \exp(x)} \right) = -\frac{e^x}{(e^x - 1)^3} (x - 2e^x + xe^x + 2),$$

thus when $x \in (-\infty, 0)$, we get

$$g''(x) \leq 0,$$

as a consequence g is convex, and

$$g(ut) \geq \frac{1}{2} (g(-\lambda t) + g((2u + \lambda)t)).$$

Therefore when $t \in (0, +\infty)$, h_1 is decrease. Consequently,

$$\begin{aligned} \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{(1 - \exp(ut))^2}{(1 - e^{\lambda t} e^{2ut})} &\leq \lim_{t \rightarrow 0} \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{(1 - \exp(ut))^2}{(1 - e^{\lambda t} e^{2ut})} \\ &= -\frac{u^2}{\lambda(2u + \lambda)}. \end{aligned}$$

Notice that when $t \in (0, +\infty)$, the inequality

$$\exp(ut) t^2 \leq (1 - \exp(ut))^2$$

holds, so

$$\frac{1}{(1 - e^{-\lambda t})} \cdot \frac{2 \exp(ut) t^2}{(1 - e^{\lambda t} e^{2ut})} \leq \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{2 (1 - \exp(ut))^2}{(1 - e^{\lambda t} e^{2ut})} \leq -\frac{2u^2}{\lambda(2u + \lambda)}.$$

On the other hand,

$$\frac{(1 - \cos vt)}{t^2} \leq \lim_{t \rightarrow 0} \frac{(1 - \cos vt)}{t^2} = \frac{1}{2} v^2,$$

thus

$$\frac{1}{(1 - e^{-\lambda t})} \cdot \frac{2 \exp(ut) (1 - \cos vt)}{(1 - e^{\lambda t} e^{2ut})} \leq -\frac{2u^2}{\lambda(2u + \lambda)} \frac{1}{2} v^2 = -u^2 \frac{v^2}{\lambda(2u + \lambda)}.$$

But

$$\lim_{t \rightarrow 0} \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{2 \exp(ut) (1 - \cos vt)}{(1 - e^{\lambda t} e^{2ut})} = -\frac{v^2}{\lambda(2u + \lambda)},$$

therefore for any $t \in (0, \infty)$, one has

$$\begin{aligned} h(\lambda, u, v, t) &\leq h(\lambda, u, v, 0) \\ &= \lim_{t \rightarrow 0} \frac{1}{(1 - e^{-\lambda t})} \cdot \frac{(1 - \exp(ut))^2 + 2 \exp(ut)(1 - \cos vt)}{(1 - e^{\lambda t} e^{2ut})} \\ &= -\frac{1}{\lambda(2u + \lambda)} (u^2 + v^2). \end{aligned}$$

■

Theorem 4.11 *Let $\psi \in LFT(\mathbb{H}^N)$ be hyperbolic with*

$$\psi(z, u, v, w) = \left(\lambda z + 2i \langle w, a \rangle + b, \sqrt{\lambda} u, \sqrt{\lambda} Dv, \sqrt{\lambda} Aw \right),$$

where $A = \text{diag}(\lambda_1, \dots, \lambda_r)$ and $|\lambda_j| < 1$ for $j = 1, 2, \dots, r$. Let

$$\lambda_j = \exp(-u_j + iv_j)$$

with $u_j > 0$ and $v_j \in [0, 2\pi)$. If

$$\text{Im } b \geq \langle \Theta a, a \rangle$$

with

$$\Theta = \text{diag} \left(\frac{\lambda - 1}{2u_1 \ln \lambda} \frac{\left(\frac{\ln \lambda}{2} + u_1\right)^2 + v_1^2}{\left|\lambda - \sqrt{\lambda} \lambda_1\right|^2}, \dots, \frac{\lambda - 1}{2u_r \ln \lambda} \frac{\left(\frac{\ln \lambda}{2} + u_r\right)^2 + v_r^2}{\left|\lambda - \sqrt{\lambda} \lambda_r\right|^2} \right),$$

then ψ can be embedded into a semigroup of \mathbb{H}^N .

Proof. Let

$$M = \text{diag}(-u_1 + iv_1, \dots, -u_r + iv_r),$$

then

$$A = \exp(M).$$

Denote by

$$\begin{aligned} \lambda_t &= \lambda^t, \\ A_t &= \exp(tM), \\ a_t &= \left(\lambda - \sqrt{\lambda} A^H \right)^{-1} \left(\lambda_t - \sqrt{\lambda_t} A_t^H \right) a, \\ b_t &= \frac{1 - \lambda_t}{1 - \lambda} b, \\ Q_t &= E - A_t^H A_t. \end{aligned}$$

Then

$$\langle Q_t^+ a_t, a_t \rangle = a^H \text{diag}(\alpha_1(t), \dots, \alpha_s(t)) a,$$

where

$$\alpha_j(t) = \frac{\left| \lambda^t - \sqrt{\lambda} e^{t(-u_j - iv_j)} \right|^2}{(1 - e^{-2tu_j}) \left| \lambda - \sqrt{\lambda} e^{-u_j - tv_j} \right|^2} = \frac{\lambda^{2t} \left| 1 - e^{-t(\frac{\ln \lambda}{2} + u_j) + iv_j} \right|^2}{(1 - e^{-2tu_j}) \left| \lambda - \sqrt{\lambda} \lambda_j \right|^2}.$$

Notice that according to Lemma 4.10, for $\frac{\ln \lambda}{2} + u_j > 0, v_j \geq 0$ and $t \geq 0$,

$$\begin{aligned} \frac{\lambda^t \left| 1 - e^{-t(\frac{\ln \lambda}{2} + u_j) + iv_j} \right|^2}{(1 - e^{-2tu_j}) (\lambda^t - 1)} &= \frac{\left| 1 - e^{-t(\frac{\ln \lambda}{2} + u_j) + iv_j} \right|}{\left(1 - e^{t \ln \lambda} e^{-2t(u_j + \frac{\ln \lambda}{2})} \right) (1 - e^{-t \ln \lambda})} \\ &\leq -\frac{1}{\ln \lambda \cdot (-2u_j)} \left[\left(\frac{\ln \lambda}{2} + u_j \right)^2 + v_j^2 \right] \\ &= \frac{1}{2u_j \ln \lambda} \left[\left(\frac{\ln \lambda}{2} + u_j \right)^2 + v_j^2 \right], \end{aligned}$$

Thus we get

$$\frac{1}{\lambda^t} \frac{(\lambda - 1)}{(\lambda^t - 1)} \langle Q_t^+ a_t, a_t \rangle \leq a^H \Theta a \leq b.$$

Our conclusion follows from Proposition 4.9. ■

4.4 The case of dimensional 2 and automorphisms

When considering our question on \mathbb{C}^2 , our conclusion would look very simple.

Let $\varphi \in LFT(B_2)$ be parabolic, then φ is conjugated to

$$\psi_1(u_1, u_2) = (u_1 + 2ibu_2 + c, \lambda u_2)$$

or

$$\psi_2(u_1, u_2) = (u_1 + c, e^{i\theta} u_2)$$

or

$$\psi_3(u_1, u_2) = (u_1 + 2iau_2 + c, u_2 + a),$$

where $a, b, c \in \mathbb{C}$ and $\lambda \in (0, 1)$. In the first case, let

$$\lambda = \exp(-\mu + iv)$$

where $\mu > 0$ and $v \in [0, 2\pi)$. If

$$\operatorname{Im} c \geq \frac{|b|^2 (\mu^2 + v^2)}{\mu |1 - \lambda|^2}.$$

Then φ can be embedding into a semigroup of \mathbb{B}_2 . Since ψ_2 and ψ_3 can always be embedded into a semigroup, in the second and the third cases, φ can always be embedded into a semigroup.

When $\varphi \in LFT(B_2)$ is hyperbolic, according to the proof of Theorem 3.6, φ is conjugated to

$$\psi_1(u_1, u_2) = (\lambda u_1 + 2i \langle u_2, b \rangle + c, \sqrt{\lambda} \alpha u_2)$$

or

$$\psi_2(u_1, u_2) = (\lambda u_1 + a, u_2 + b).$$

Let $\alpha = e^{\beta + i\gamma}$. ψ_1 can be embedded into a semigroup of \mathbb{H}^2 if

$$\operatorname{Im} c \geq \frac{\lambda - 1}{2\beta \ln \lambda} \frac{\left(\frac{\ln \lambda}{2} + \beta\right)^2 + \gamma^2}{\left|\lambda - \sqrt{\lambda} \alpha\right|^2} |b|^2.$$

When concern about ψ_2 , we have the following theorem:

Theorem 4.12 *Let $\psi \in LFT(\mathbb{H}^2)$ with*

$$\psi(u_1, u_2) = (\lambda u_1 + a, u_2 + b).$$

If

$$\operatorname{Im} a \geq \frac{(\lambda - 1)}{\ln^2 \lambda} |b|^2,$$

then ψ can be embedded into a semigroup of \mathbb{H}^2 .

Proof. It is easy to verify that

$$\frac{d}{dt} \left(\frac{\lambda^t - 1}{t} \right) = \frac{1}{t^2} (t\lambda^t \ln \lambda - \lambda^t + 1)$$

and

$$\frac{d}{dt} (t\lambda^t \ln \lambda - \lambda^t + 1) = t\lambda^t \ln^2 \lambda,$$

thus for any $t > 0$, we obtain

$$\frac{\lambda^t - 1}{t} \geq \lim_{t \rightarrow 0} \frac{\lambda^t - 1}{t} = \ln \lambda.$$

As a consequence,

$$\frac{t^2}{(\lambda^t - 1)^2} \leq \frac{1}{\ln^2 \lambda}.$$

Let

$$\psi_t(u_1, u_2) = \left(\lambda^t u_1 + \frac{\lambda^t - 1}{\lambda - 1} a, u_2 + tb \right),$$

For any $t \geq 0$,

$$\begin{aligned} \operatorname{Im} \left(\frac{\lambda^t - 1}{\lambda - 1} a \right) &\geq \frac{\lambda^t - 1}{\lambda - 1} \cdot \frac{(\lambda - 1)}{\ln^2 \lambda} |b|^2 \\ &\geq |tb|^2. \end{aligned}$$

According to Theorem 3.6, ψ_t is a self-map of \mathbb{H}^2 and thus $\{\psi_t\}$ is a semigroup of \mathbb{H}^2 . Hence ψ can be embedded into a semigroup of \mathbb{H}^2 . ■

When φ is an elliptic automorphism, φ is conjugated to a unitary transformation of B_N and therefore φ can always be embedded into a semigroup of B_N . In the non-elliptic case, if φ is a parabolic automorphism with at least one invariant slice, then by Theorem 4.3 of [5], there exists a unitary matrix U , such that φ is conjugated to

$$\psi(u', u'') = (z' + ic, U z''),$$

where α is the boundary dilation coefficient of φ , $c \in \mathbb{R}$, thus by Theorem 4.6, φ can be embedded into a semigroup. If φ is a hyperbolic automorphism, then by Proposition 3.9, φ is conjugated to

$$\psi_1(u', u'') = \frac{1}{\alpha} (z' + ic, \sqrt{\alpha} U z''),$$

where U is a unitary matrix, thus by Theorem 4.9, φ can be embedded into a semigroup of B_N . Together with Lemma 4.7, we obtain

Corollary 4.13 *Let φ be an automorphism of B_N , then φ can be embedded into a semigroup of B_N .*

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